Received: *April 14, 2000,* Accepted: *May 26, 2000*

Small instanton transitions in heterotic M-theory

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Abstract: We discuss non-perturbative phase transitions, within the context of heterotic M-theory, which occur when all, or part, of the wrapped five-branes in the five-dimensional bulk space come into direct contact with a boundary brane. These transitions involve the transformation of the five-brane into a "small instanton" on the Calabi-Yau space at the boundary brane, followed by the "smoothing out" of the small instanton into a holomorphic vector bundle. Small instanton phase transitions change the number of families, the gauge group or both on the boundary brane, depending upon whether a base component, fiber component or both components of the five-brane class are involved in the transition. We compute the conditions under which a small instanton phase transition can occur and present a number of explicit, phenomenologically relevant examples.

Keywords: M-Theory, Superstring Vacua.

JHEP05(2000)045

Contents

1. Introduction

In fundamental work, Hořava and Witten [1, 2] showed that chiral fermions can be obtained from M-theory by compactifying $D = 11$, $N = 1$ supergravity on an $S^1/\mathbb{Z}_2$ orbifold. The resultant theory consists of an eleven-dimensional "bulk" space with two ten-dimensional boundary fixed planes, one at one end of the bulk space and one at the other. Furthermore, in order for the theory to be anomaly free, these authors showed that there must be a $D = 10$, $N = 1$ E_8 Yang-Mills supermultiplet on each of the boundary planes. Witten [3] further demonstrated that, when compactified on a Calabi-Yau manifold, realistic low energy parameters, such as the $D = 4$ Newton's constant, will occur only if the the orbifold radius is substantially larger than the radius of the Calabi-Yau threefold.

Based on this observation, the effective five-dimensional theory arising from the compactification of $D = 11$, $N = 1$ supergravity on a Calabi-Yau threefold was constructed [4, 5]. This five-dimensional regime of M-theory was shown to be described by a specific type of gauged $D = 5$, $N = 1$ supergravity coupled to hyper and vector supermultiplets in the bulk space and to gauge and matter supermultiplets on the two four-dimensional boundary planes. The gauge symmetry introduces "cosmological" potentials for hyperscalars of a very specific type, both in the bulk space and on the two boundary planes. These potentials have exactly the right form so as to support BPS three-branes as solutions of the equations of motion. It was shown in [4] that the minimal static vacuum of this theory consists of two BPS three-branes, one located at one boundary plane and one at the other. However, as discussed in [6], more complicated "non-perturbative" vacua are possible which, in addition to the pair of boundary three-branes, allow for one or more five-branes in the bulk space. These five-branes are wrapped on holomorphic curves in the background Calabi-Yau threefold. When compactifying, it is necessary to specify the $N = 1$ supersymmetry preserving E_8 gauge "instanton" localized on the Calabi-Yau space at each boundary three-brane. Such gauge configurations satisfy the hermitean Yang-Mills equations. These instanton vacua are not arbitrary, being required to be, among other things, consistent with anomaly cancellation. The physical effect of a non-trivial gauge instanton is to break E_8 to a smaller gauge group, as well as to introduce chiral "matter" superfields on the associated boundary three-brane. It was shown in [7, 8] that hermitean Yang-Mills instantons can be described in terms of smooth, semi-stable, holomorphic vector bundles. Using, and extending, mathematical techniques introduced in [9, 10, 11, 12], it was demonstrated in a series of papers [13, 14, 15] that there are phenomenologically relevant, anomaly free vacuum solutions. These vacua have three-families of quarks and leptons, as well as realistic grand unified groups, such as SO(10) and SU(5), or the standard gauge group, $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, on one of the boundary three-branes. This brane is called the "observable" brane. For simplicity, we usually take the vector bundle to be trivial on the other boundary three-brane, which, thus, has unbroken gauge group E_8. This is called the "hidden" brane. Generically, we find that such vacua contain additional five-branes "living" in the bulk space and wrapped on holomorphic curves in the background Calabi-Yau threefold.

To conclude, we have shown in [4, 5, 6] that a fundamental "brane world" emerges from M-theory compactified on a Calabi-Yau threefold and an $S^1/\mathbb{Z}_2$ orbifold. Typically, the Calabi-Yau radius is of the order inverse 10^{16} GeV, whereas the orbifold radius can be anywhere from an order of magnitude larger to inverse 10^{12} GeV [16]. For length scales between these two radii, this world consists of a five-dimensional $N = 1$ supersymmetric bulk space bounded on two sides by BPS three-branes. One of these boundary three-branes, the "observable" brane, has a realistic gauge group and matter content. The other boundary three-brane is the "hidden" brane which, in this paper, we will choose to have unbroken E_8 gauge group. In addition, there generically are wrapped five-branes "living" in the bulk space. We refer to this five-dimensional "brane world" as heterotic M-theory.

A wrapped BPS five-brane in the bulk space has a modulus corresponding to translation of the five-brane in the orbifold direction. An important question to ask is: What happens to a wrapped bulk five-brane in heterotic M-theory when it is translated across the bulk space and comes into direct contact with one of the boundary three-branes? This is the question that we address in this paper. For specificity, and because it is physically more interesting, we will consider "collisions" of a bulk five-brane with the "observable" boundary three-brane. All of our results, however, apply equally well to collisions with the "hidden" brane. We will show the following. Upon contact with the boundary three-brane, the wrapped five-brane disappears and its data is "absorbed" into a singular "bundle", called a torsion free sheaf, localized on the Calabi-Yau threefold associated with that boundary three-brane. This singular, torsion free sheaf is referred to as a "small instanton" [17]. This small instanton can then be "smoothed" out by moving in its moduli space to a smooth holomorphic vector bundle. The physical picture is that the bulk five-brane disappears, thus altering the instanton vacuum on the boundary three-brane. The altered gauge vacuum generically has different topological data than the instanton prior to the five-brane "collision" with the boundary brane. In particular, the third Chern class of the associated vector bundle can change, thus changing the number of quark and lepton families on the observable wall. That is, the vacuum can undergo a "chirality-changing" phase transition. Furthermore, the structure group of the vector bundle can change, thus altering the unbroken gauge group on the boundary brane. That is, the vacuum can undergo a "gauge-changing" phase transition. Whether the phase transition is chirality-changing, gauge-changing, or both depends on the topological structure of the bulk five-brane being absorbed. At least for the smooth holomorphic vector bundles discussed in this paper, we find that chirality-changing small instanton phase transitions only occur for specific initial topological data, and are otherwise obstructed. Gauge-changing transitions can always occur. All of the work presented in this paper is within the context of compactification on elliptically fibered Calabi-Yau threefolds. Related discussions involving "monad" Calabi-Yau spaces were given in [18].

Specifically, in this paper we do the following. In section 2, we review some properties of elliptically fibered Calabi-Yau threefolds that we will use in our discussion. In section 3, we present an outline of the spectral cover construction of smooth, stable (and, hence, semi-stable) holomorphic SU(n) vector bundles over elliptically fibered Calabi-Yau threefolds, and give some explicit examples. Section 4 is devoted to discussing the two fundamental topological conditions that must be satisfied in any realistic particle physics vacuum. These conditions are associated with the requirements of anomaly cancellation and three-families of quarks and lep-

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tons. Chirality-changing small instanton phase transitions are explicitly constructed in section 5. It is shown that these transitions are associated with "absorbing" all, or part, of the base component of the bulk space five-brane class. The mathematical structure of the associated small instanton is presented and a detailed discussion of the conditions under which it can be "smoothed" to a vector bundle is given. We compute the Chern classes both before and after the small instanton transition and give an explicit formula for calculating the change in the number of families. We present several explicit examples, one transition involving the complete base component and the second involving only a portion of the base component of the five-brane class. In section 6, we discuss gauge-changing small instanton transitions. It is shown that these transitions are associated with "absorbing" all, or part, of the pure fiber component of the bulk space five-brane class. We present the mathematical structure of the associated small instanton and show that this can always be "smoothed" to a reducible, semi-stable vector bundle. We construct the Chern classes both before and after the small instanton transition and show that the number of families is unchanged. It is demonstrated , however, that the structure group of the vector bundle changes from $\mathrm{SU}(n)$ to $\mathrm{SU}(n) \times \mathrm{SU}(m)$ for restricted values of m. Hence, the unbroken gauge group on the observable brane, which is the commutant of the structure group, also changes in a calculable way. We present an explicit example of this type of transition. Finally, in section 7, we present our conclusions.

2. Elliptically fibered Calabi-Yau threefolds

In this paper, we will consider Calabi-Yau threefolds, X, that are structured as elliptic curves fibered over a base surface, B. Specifically, there is a mapping $\pi : X \to B$ such that $\pi^{-1}(b)$ is a torus, E_b, for each point $b \in B$. We further require that this torus fibered threefold has a zero section; that is, there exists an analytic map $\sigma : B \to X$ that assigns to every element b of B, an element $\sigma(b) \in E_b$. The point $\sigma(b)$ acts as the zero element for the group law and turns E_b into an elliptic curve.

A simple representation of an elliptic curve is given in the projective space $\mathbb{CP}^2$ by the Weierstrass equation

$$zy^2 = 4x^3 - g_2 x z^2 - g_3 z^3 \,, \tag{2.1}$$

where (x, y, z) are the homogeneous coordinates of $\mathbb{CP}^2$ and g_2, g_3 are constants. The origin of the elliptic curve is located at $(x, y, z) = (0, 1, 0)$. Note that near the origin $z \cong 4x^3$ and, hence, has a third order zero as $x \longrightarrow 0$. This same equation can represent the elliptic fibration, X, if the coefficients g_2, g_3 in the Weierstrass equation are functions over the base surface, B. The correct way to express this globally is to replace the projective plane $\mathbb{CP}^2$ by a $\mathbb{CP}^2$-bundle $P \to B$ and then require that g_2, g_3 be sections of appropriate line bundles over the base. If we denote the conormal bundle to the zero section $\sigma(B)$ by $\mathcal{L}$, then $P = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$, where $\mathbb{P}(M)$ stands for the projectivization of a vector bundle M. There is a hyperplane line bundle $\mathcal{O}_P(1)$ on P which corresponds to the divisor $\mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^3) \subset P$ and the coordinates x, y, z are sections of $\mathcal{O}_P(1) \otimes \mathcal{L}^2, \mathcal{O}_P(1) \otimes \mathcal{L}^3$ and $\mathcal{O}_P(1)$ respectively. It then follows from (2.1) that the coefficients g_2 and g_3 are sections of $\mathcal{L}^4$ and $\mathcal{L}^6$.

It will be useful in this paper to define new coordinates,[1] $\mathbf{X}$,$\mathbf{Y}$,$\mathbf{Z}$, on X by $x = \mathbf{XZ}$, $y = \mathbf{Y}$ and $z = \mathbf{Z}^3$.

It follows that $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are now sections of line bundles

$$\mathbf{X} \sim \mathcal{O}(2\sigma) \otimes \mathcal{L}^2\,, \qquad \mathbf{Y} \sim \mathcal{O}(3\sigma) \otimes \mathcal{L}^3\,, \qquad \mathbf{Z} \sim \mathcal{O}(\sigma) \tag{2.2}$$

respectively. The coefficients g_2 and g_3 remain sections of line bundles

$$g_2 \sim \mathcal{L}^4\,, \qquad g_3 \sim \mathcal{L}^6\,. \tag{2.3}$$

The symbol "$\sim$" simply means "section of".

The requirement that elliptically fibered threefold, X, be a Calabi-Yau space constrains the first Chern class of the tangent bundle, TX, to vanish. That is,

$$c_1(TX) = 0\,. \tag{2.4}$$

It follows from this that

$$\mathcal{L} = K_B^{-1}\,, \tag{2.5}$$

where K_B is the canonical bundle on the base, B. Condition (2.5) is rather strong and restricts the allowed base spaces of an elliptically fibered Calabi-Yau threefold to be del Pezzo, Hirzebruch and Enriques surfaces, as well as certain blow-ups of Hirzebruch surfaces [19, 20].

3. Spectral cover description of SU(n) vector bundles

As discussed in detail in [9, 14], SU(n) vector bundles over an elliptically fibered Calabi-Yau threefold can be explicitly constructed from two mathematical objects, a divisor $\mathcal{C}$ of X, called the spectral cover, and a line bundle $\mathcal{N}$ on $\mathcal{C}$. Let us discuss the relevant properties of each in turn. In this section, we will describe only stable SU(n) vector bundles constructed from irreducible spectral covers. Semi-stable vector bundles associated with reducible spectral covers will be discussed in section 6.

[1] To see that such coordinates exist, note that the cubic behavior of z as $x \to 0$ implies that the restriction of $\mathcal{O}_P(1)$ to X is precisely the line bundle $\mathcal{O}_X(3\sigma)$. Therefore, we may take $\mathbf{Z}$ to be the unique (up to scalar) section of $\mathcal{O}(\sigma)$ and normalize it so that $z = \mathbf{Z}^3$.

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3.1 Spectral cover

A spectral cover, $\mathcal{C}$, is a surface in X that is an n-fold cover of the base B. That is, $\pi_{\mathcal{C}} : \mathcal{C} \to B$. The general form for a spectral cover is given by

$$\mathcal{C} = n\sigma + \pi^*\eta\,, \tag{3.1}$$

where σ is the zero section and η is some curve in the base B. The terms in (3.1) can be considered either as elements of the homology group $H_4(X, \mathbb{Z})$ or, by Poincare duality, as elements of cohomology $H^2(X, \mathbb{Z})$. This ambiguity will occur in many of the topological expressions in this paper.

In terms of the coordinates $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ introduced above, it can be shown that the spectral cover can be represented as the zero set of the polynomial

$$s = a_0\mathbf{Z}^n + a_2\mathbf{X}\mathbf{Z}^{n-2} + a_3\mathbf{Y}\mathbf{Z}^{n-3} + \cdots + a_n\mathbf{X}^{n/2}\,, \tag{3.2}$$

for n even and ending in $a_n\mathbf{X}^{\frac{n-3}{2}}\mathbf{Y}$ if n is odd, along with the relations (2.2). This tells us that the polynomial s must be a holomorphic section of the line bundle of the spectral cover, $\mathcal{O}(\mathcal{C})$. That is,

$$s \sim \mathcal{O}(n\sigma + \pi^*\eta)\,. \tag{3.3}$$

It follows from this and equations (2.2) and (2.3), that the coefficients a_i in the polynomial s must be sections of the line bundles

$$a_i \sim \pi^* K_B^i \otimes \mathcal{O}(\pi^*\eta) \tag{3.4}$$

for $i = 1, \ldots, n$ where we have used expression (2.5).

In order to describe vector bundles most simply, there are two properties that we require the spectral cover to possess. The first, which is shared by all spectral covers, is that

- $\mathcal{C}$ must be an effective class in $H_4(X, \mathbb{Z})$.

This property is simply an expression of the fact the spectral cover must be an actual surface in X. It can easily be shown that

$$\mathcal{C} \subset X \text{ is effective } \iff \eta \text{ is an effective class in } H_2(B, \mathbb{Z}). \tag{3.5}$$

The second property that we require for the spectral cover is that

- $\mathcal{C}$ is an irreducible surface.

This condition is imposed because it guarantees that the associated vector bundle is stable. It is important to note, however, that semi-stable vector bundles can be constructed from reducible spectral covers, as we will do later in this paper.

Deriving the conditions under which $\mathcal{C}$ is irreducible is not completely trivial and will be discussed in detail elsewhere [21]. Here, we will simply state the results. First, recall from (3.4) that $a_i \sim \pi^* K_B^i \otimes \mathcal{O}(\pi^*\eta)$ and, hence, the zero locus of a_i is a divisor, $D(a_i)$, in X. Then, we can show that $\mathcal{C}$ is an irreducible surface if

$$D(a_0) \text{ is an irreducible divisor in } X \tag{3.6}$$

and

$$D(a_n) \text{ is an effective class in } H_4(X, \mathbb{Z})\,. \tag{3.7}$$

Using Bertini's theorem, it can be shown that condition (3.6) is satisfied if the linear system $|\eta|$ is base point free. "Base point free" means that for any $b \in B$, we can find a member of the linear system $|\eta|$ that does not pass through the point b.

In order to make these concepts more concrete, we take, as an example, the base surface to be

$$B = \mathbb{F}_r \tag{3.8}$$

and derive the conditions under which (3.5), (3.7) and (3.6) are satisfied. Recall[14] that the homology group $H_2(\mathbb{F}_r, \mathbb{Z})$ has as a basis the effective classes $\mathcal{S}$ and $\mathcal{E}$ with intersection numbers

$$\mathcal{S}^2 = -r\,, \qquad \mathcal{S} \cdot \mathcal{E} = 1\,, \qquad \mathcal{E}^2 = 0\,. \tag{3.9}$$

Then, in general, $\mathcal{C}$ is given by expression (3.1) where

$$\eta = a\mathcal{S} + b\mathcal{E} \tag{3.10}$$

and a,b are integers. One can easily check that η is an effective class in $\mathbb{F}_r$, and, hence, that $\mathcal{C}$ is an effective class in X, if and only if

$$a \geq 0\,, \qquad b \geq 0\,. \tag{3.11}$$

It is also not too hard to demonstrate that the linear system $|\eta|$ is base point free if and only if

$$b \geq ar\,. \tag{3.12}$$

Imposing this constraint then implies that $D(a_0)$ is an irreducible divisor in X. Finally, we can show that for $D(a_n)$ to be effective in X one must have

$$a \geq 2n\,, \qquad b \geq n(r+2)\,. \tag{3.13}$$

We will give a detailed derivation of (3.12) and (3.13) elsewhere [21]. Combining conditions (3.12) and (3.13) then guarantees that $\mathcal{C}$ is an irreducible surface. To be even more specific, we now present two physically relevant examples of this type.

Example 1. Choose the structure group of the vector bundle to be

$$G = \mathrm{SU}(5)\,. \tag{3.14}$$

Hence, $n = 5$. Also, restrict $r = 1$. Now choose

$$\eta = 12\mathcal{S} + 15\mathcal{E} \tag{3.15}$$

so that $a = 12$ and $b = 15$. These parameters are easily shown to satisfy the conditions (3.11), (3.12) and (3.13). Therefore, the associated spectral surface

$$\mathcal{C} = 5\sigma + \pi^*(12\mathcal{S} + 15\mathcal{E}) \tag{3.16}$$

is both effective and irreducible.

Example 2. As a second example, consider again

$$G = \mathrm{SU}(5)\,. \tag{3.17}$$

Hence, $n = 5$. Again, restrict $r = 1$. Now choose

$$\eta = 24\mathcal{S} + 36\mathcal{E} \tag{3.18}$$

so that $a = 24$ and $b = 36$. These parameters also satisfy equations (3.11), (3.12) and (3.13). Therefore, the associated spectral surface

$$\mathcal{C} = 5\sigma + \pi^*(24\mathcal{S} + 36\mathcal{E}) \tag{3.19}$$

is again both effective and irreducible.

The reason for choosing these two examples will become clear soon. We now turn to the second mathematical object that is required to specify an $\mathrm{SU}(n)$ vector bundle.

3.2 The line bundle $\mathcal{N}$

As discussed in [9, 14], in addition to the spectral cover it is necessary to specify a line bundle, $\mathcal{N}$, over $\mathcal{C}$. For $\mathrm{SU}(n)$ vector bundles, this line bundle must be a restriction of a global line bundle on X (which we will again denote by $\mathcal{N}$), satisfying the condition

$$c_1(\mathcal{N}) = n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi^*\eta + \left(\frac{1}{2} + n\lambda\right)\pi^* c_1(B)\,, \tag{3.20}$$

where $c_1(\mathcal{N})$, $c_1(B)$ are the first Chern classes of $\mathcal{N}$ and B respectively and λ is, a priori, a rational number. Since $c_1(\mathcal{N})$ must be an integer class, it follows that either

$$n \quad \text{is odd}\,, \qquad \lambda = m + \frac{1}{2} \tag{3.21}$$

or

$$n \quad \text{is even}\,, \qquad \lambda = m\,, \qquad \eta = c_1(B) \bmod 2 \tag{3.22}$$

where $m \in \mathbb{Z}$. In this paper, for simplicity, we will always take examples where n is odd.

3.3 SU(n) vector bundle

Given a spectral cover, $\mathcal{C}$, and a line bundle, $\mathcal{N}$, satisfying the above properties, one can now uniquely construct an SU(n) vector bundle, V. This can be accomplished in two ways. First, as discussed in [9, 14], the vector bundle can be directly constructed using the associated Poincare bundle, $\mathcal{P}$. The result is that

$$V = \pi_{1*}(\pi_2^* \mathcal{N} \otimes \mathcal{P}) \,, \tag{3.23}$$

where π_1 and π_2 are the two projections of the fiber product $X \times_B \mathcal{C}$ onto the two factors X and $\mathcal{C}$. We refer the reader to [9, 14] for a detailed discussion. Equivalently, V can be constructed directly from $\mathcal{C}$ and $\mathcal{N}$ using the Fourier-Mukai transformation, as discussed in [15, 21, 22]. Both of these constructions work in reverse, yielding the spectral data $(\mathcal{C}, \mathcal{N})$ up to the overall factor of K_B given the vector bundle V. Throughout this paper we will indicate this relationship between the spectral data and the vector bundle by writing

$$(\mathcal{C}, \mathcal{N}) \longleftrightarrow V \,. \tag{3.24}$$

The Chern classes for the SU(n) vector bundle V have been computed in [9] and [15, 23]. The results are

$$c_1(V) = 0 \tag{3.25}$$

since $\operatorname{tr} F = 0$ for the structure group SU(n),

$$c_2(V) = \eta\sigma - \frac{1}{24} c_1(B)^2(n^3 - n) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right) n\eta(\eta - nc_1(B)) \tag{3.26}$$

and

$$c_3(V) = 2\lambda\sigma\eta(\eta - nc_1(B)) \,. \tag{3.27}$$

In order to make these concepts more concrete, we again take the base surface to be

$$B = \mathbb{F}_r \,. \tag{3.28}$$

The Chern classes for this surface are known and given by

$$c_1(\mathbb{F}_r) = 2\mathcal{S} + (r+2)\mathcal{E} \,, \qquad c_2(\mathbb{F}_r) = 4 \,. \tag{3.29}$$

Now consider the two specific examples discussed above.

Example 1. In this example, the structure group of the vector bundle is chosen to be $G = \mathrm{SU}(5)$ and, hence, $n = 5$. Also, we restrict $r = 1$ and take $\eta = 12\mathcal{S} + 15\mathcal{E}$. We further specify the line bundle, $\mathcal{N}$, by choosing

$$\lambda = \frac{1}{2} \,. \tag{3.30}$$

Note that this requirement is consistent with condition (3.21) since $n = 5$ is odd. It follows from (3.9), (3.26) and (3.29) that

$$c_2(V) = (12\mathcal{S} + 15\mathcal{E})\sigma - 40F\,, \tag{3.31}$$

where F is the generic class of the fiber, and from (3.9), (3.27) and (3.29) that

$$c_3(V) = 6\,. \tag{3.32}$$

Example 2. As a second example, consider again $G = \mathrm{SU}(5)$, $n = 5$, $r = 1$ and we take $\eta = 24S + 36\mathcal{E}$. Further specify the line bundle, $\mathcal{N}$, by choosing

$$\lambda = \frac{1}{2}\,. \tag{3.33}$$

It follows from (3.9), (3.26) and (3.29) that

$$c_2(V) = (24\mathcal{S} + 36\mathcal{E})\sigma - 40F \tag{3.34}$$

and from (3.9), (3.27) and (3.29) that

$$c_3(V) = 672\,. \tag{3.35}$$

To conclude, in this section we have discussed the construction and properties of stable $\mathrm{SU}(n)$ vector bundles associated with irreducible spectral covers. For the remainder of this paper, for brevity, we will refer to such bundles simply as "stable $\mathrm{SU}(n)$ vector bundles".

4. Physical topological conditions

As discussed in a number of papers [6, 23, 24, 25], there are two fundamental conditions that must be satisfied in any physically acceptable heterotic M-theory.

4.1 Anomaly cancellation

The first condition is a direct consequence of demanding that the theory be anomaly free. This requires that the Bianchi identity for the field strength of the three-form be modified by the addition of both gauge and tangent bundle Chern classes as well as by sources for possible bulk space five-branes. Integrating this modified Bianchi identity over an arbitrary four-cycle then gives the topological condition

$$c_2(V_1) + c_2(V_2) + W = c_2(TX)\,,$$

where V_1 and V_2 are the vector bundles on the observable and hidden boundary branes respectively and TX is the tangent bundle of the Calabi-Yau threefold X. In addition, W is the class of the holomorphic curve in X around which possible bulk

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space five-branes are wrapped. Any anomaly free heterotic M-theory must satisfy this condition. In this paper, for simplicity, we will always choose the vector bundle on the hidden sector, V_2, to be trivial. This ensures an unbroken E_8 gauge group in the hidden sector and simplifies equation (4.1) to

$$W = c_2(TX) - c_2(V)\,, \tag{4.1}$$

where we now denote V_1 by V. Given the second Chern classes for V and TX, this equation acts as a definition for the five-brane class W. As such, it is not, a priori, a constraint. However, the five-brane class must, on physical grounds, be represented by an actual surface in X. Hence,

$$W \text{ must be an effective class in } H_2(X, \mathbb{Z})\,. \tag{4.2}$$

This condition puts a non-trivial constraint on the choice of the vector bundle V. It is important to note, however, that the constraint of effectiveness of W is much less restrictive then trying to set $c_2(V) = c_2(TX)$, as we will see.

Using equation (3.26) and the fact that

$$c_2(TX) = (c_2(B) + 11c_1(B)^2)F + 12\sigma \cdot c_1(B)\,, \tag{4.3}$$

where $c_2(B)$ is the second Chern class of the base B, it follows from (4.1) that

$$W = W_B\sigma + a_f F\,, \tag{4.4}$$

where

$$W_B = \pi^*(12c_1(B) - \eta) \tag{4.5}$$

and

$$a_f = c_2(B) + \left(11 + \frac{n(n^2-1)}{24}\right) c_1(B)^2 - \frac{n}{2}\left(\lambda^2 - \frac{1}{4}\right)\eta(\eta - nc_1(B))\,. \tag{4.6}$$

For concreteness, let us evaluate these quantities for the two sample cases discussed above.

Example 1. In this case, $G = \mathrm{SU}(n)$, $n = 5$, $B = \mathbb{F}_1$, $\eta = 12\mathcal{S} + 15\mathcal{E}$ and $\lambda = 1/2$. It then follows from (3.9), (3.29), (4.5) and (4.6) that

$$W_B = 12\mathcal{S} + 21\mathcal{E}\,, \qquad a_f = 132\,. \tag{4.7}$$

Example 2. Here $G = \mathrm{SU}(n)$, $n = 5$, $B = \mathbb{F}_1$, $\eta = 24\mathcal{S} + 36\mathcal{E}$ and $\lambda = 1/2$. It then follows from (3.9), (3.29), (4.5) and (4.6) that

$$W_B = 0, \qquad a_f = 132\,. \tag{4.8}$$

These examples elucidate two important properties of five-brane classes W within the context of the stable SU(n) vector bundles discussed so far. The first property is that it is possible to choose vector bundles such that $W_B = 0$, as in example 2. We will show in the next section that, under certain conditions, a vacuum with a five-brane class with non-vanishing base component, $W_B \neq 0$, can make a phase transition via a "small instanton" to a vacuum in which the five-brane class base component vanishes, $W_B = 0$. Note, however, that in both of the above examples the five-brane fiber component, $a_f F$, is non-zero. This turns out to indicate the second property of five-brane classes. That is, it is never possible, within context of the stable SU(n) vector bundles arising from irreducible spectral surfaces, to choose a vector bundle such that the entire five-brane class vanishes, $W = 0$. This statement is sufficiently important for us to provide a short proof.

4.2 Search for $c_2(V) = c_2(TX)$

In order for $W = 0$, it is necessary that W_B in (4.5) and a_f in (4.6) both vanish. Clearly, $W_B = 0$ requires that one choose

$$\eta = 12c_1(B)\,. \tag{4.9}$$

Inserting this expression into (4.6), we find that a_f will vanish if and only if

$$\lambda = \pm\sqrt{\frac{c_2 + \left(11 + \frac{n(n^2-1)}{24} + \frac{3n(12-n)}{2}\right)c_1^2}{6n(12-n)}}\,. \tag{4.10}$$

Recall that for $c_1(\mathcal{N})$ to be an integer class, the parameter λ must be a rational number. The square root of a rational number is generically not rational itself and, hence, we do not expect λ in (4.10) to be rational. We have checked for del Pezzo, Hirzebruch and Enriques bases with physically sensible value of n and found that this is indeed the case. We conclude, therefore, that one can never have $W = 0$ or, equivalently, that one can never set $c_2(V) = c_2(TX)$ within context of the stable SU(n) vector bundles discussed so far.

4.3 Number of generations

The second fundamental topological condition is the statement that the number of families of quarks and leptons on the observable brane is given by

$$N_{\text{gen}} = \frac{c_3(V)}{2}\,.$$

It follows from (3.27) that

$$N_{\text{gen}} = \lambda\sigma\eta(\eta - nc_1(B))\,. \tag{4.11}$$

For concreteness, let us evaluate the number of generations for the two sample cases discussed above.

Example 1. In this case, $G = \mathrm{SU}(n)$, $n = 5$, $B = \mathbb{F}_1$, $\eta = 12\mathcal{S} + 15\mathcal{E}$ and $\lambda = 1/2$. It then follows from (3.9), (3.29) and (4.11) that

$$N_{\mathrm{gen}} = 3\,. \tag{4.12}$$

Example 2. In this example $G = \mathrm{SU}(n)$, $n = 5$, $B = \mathbb{F}_1$, $\eta = 24\mathcal{S} + 36\mathcal{E}$ and $\lambda = 1/2$. It then follows from (3.9), (3.29) and (4.11) that

$$N_{\mathrm{gen}} = 336\,. \tag{4.13}$$

Once again, these examples are indicative of an important property of phase transitions via "small instantons" that change a five-brane class with $W_B \neq 0$ into a five-brane class with $W_B = 0$. That is, in any such transition the number of generations will change. Thus, such processes correspond to "chirality-changing" phase transitions.

5. Chirality-changing small instanton transitions

In this section, we will show that, within the context of the stable $\mathrm{SU}(n)$ vector bundles discussed so far, chirality-changing phase transitions via small instantons can occur. These transitions have the property that they take a vacuum with a non-zero base component, $W_B \neq 0$, in the five-brane class and transform it to a vacuum with a five-brane class with either a vanishing base component, $W_B = 0$, or a smaller base component. Such transitions do not affect the fiber component, $a_f F$, of the five-brane curve, which is non-vanishing and identical on both sides of the small instanton transition. Phase transitions which change the fiber component of W will be discussed in the next section.

Let us begin on the observable boundary brane with a stable $\mathrm{SU}(n)$ vector bundle V specified by the spectral cover $\mathcal{C}$ which is a smooth, irreducible surface in the homology class

$$\mathcal{C} = n\sigma + \pi^*\eta \tag{5.1}$$

and the line bundle $\mathcal{N}$ over $\mathcal{C}$ whose first Chern class satisfies (3.20). It follows from the above discussion that the bulk space five-brane class $W = W_B\sigma + a_f F$ does not vanish. In addition, we will demand that V and TX be such that the base component

$$W_B \neq 0\,. \tag{5.2}$$

The fiber component, $a_f F$, also does not vanish, but will not concern us in this section. Since any W in X is a four-form, it follows from Poincare duality that W_B must be a surface in X. From (4.5), we see that W_B is of the form $W_B = \pi^* z$ where $z = 12c_1(B) - \eta$.

Let us now move the bulk five-brane to the observable boundary brane and attempt to "absorb" the $\pi^* z$ part of the five-brane class into the vector bundle.

A full chirality-changing phase transition will occur if we choose $z = 12c_1(B) - \eta$. However, a "partial" transition can occur when z is any effective subcurve that splits off from $12c_1(B) - \eta$. In either case, this will result in a "bundle", $\widetilde{V}$, whose spectral cover, $\widetilde{\mathcal{C}}$, is reducible and of the form

$$\widetilde{\mathcal{C}} = \mathcal{C} \cup \pi^* z \,. \tag{5.3}$$

One might try to specify the line bundle $\widetilde{\mathcal{N}}$ over $\widetilde{\mathcal{C}}$ directly and then to construct the "bundle" $\widetilde{V}$ from this spectral data via the Fourier-Mukai transformation. However, we find it expedient to first discuss the properties of $\widetilde{V}$ and then use these properties to derive the general form of $\widetilde{\mathcal{N}}$. To do this, we begin by employing the Fourier-Mukai transformation to construct V from the spectral data $(\mathcal{C}, \mathcal{N})$. That is,

$$(\mathcal{C}, \mathcal{N}) \longrightarrow V \,, \tag{5.4}$$

where V is the original rank n vector bundle over X. Second, we must specify a line bundle ℓ on the surface $\pi^* z$ which we will glue together with $\mathcal{N}$ to produce $\widetilde{\mathcal{N}}$. A priori, ℓ is arbitrary, but, as we will see, it is actually subject to strong constraints. To find these constraints, we first use the Fourier-Mukai transformation to construct a vector bundle V_z from the spectral data $(\pi^* z, \ell)$. That is,

$$(\pi^* z, \ell) \longrightarrow V_z \,, \tag{5.5}$$

where V_z is a rank 1 vector bundle over the curve $\tilde{z} = \sigma \cdot \pi^* z$.[2] Now, the fact that, by construction, the base component of the bulk five-brane class associated with $\widetilde{V}$ must be equal to $W_B - \pi^* z$, implies that $\ell = \pi^* L$ for some line bundle L on the curve z in the base. A simple Fourier-Mukai calculation shows that V_z is just

$$V_z = i_*(L \otimes K_B) \,, \tag{5.6}$$

where i is the embedding $i : \tilde{z} \to X$ of the curve $\tilde{z}$ in X.

Given V and V_z, we now attempt to "weave" them together by relating them on the curve $\tilde{z}$ in X where their data overlap. This is done by specifying a surjection

$$\xi : V|_{\tilde{z}} \longrightarrow V_z \,. \tag{5.7}$$

That such a surjection exists and is unique will become clear shortly. Given this relation, one can define a "bundle" $\widetilde{V}$ on X via the exact sequence[3]

$$0 \to \widetilde{V} \to V \to V_z \to 0 \,. \tag{5.8}$$

[2]As a technical aside, we note that the vector bundle V_z over the curve $\tilde{z}$ can be formally extended over the Calabi-Yau threefold X as an object that vanishes everywhere outside the curve $\tilde{z}$ and is identical to V_z on $\tilde{z}$. This extended object will also be denoted by V_z. We will let context dictate which of these notions is to be used. This remark applies to all of the line bundles discussed throughout this paper.

[3]The bundle $\widetilde{V}$ defined in this way bears a special name. It is a called a Hecke transform of V and the pair (ξ, V_z) is called the center of the Hecke transform.

It can be shown that because

$$\text{codimension}\,\tilde{z} = 2 > 1\,, \tag{5.9}$$

then $\tilde{V}$ is a singular object, called a torsion free sheaf,[4] but is not a smooth vector bundle. Hence, to complete our construction, we will have to show that $\widetilde{V}$ can be "smoothed out" to a stable vector bundle. Before doing this, however, we must first compute the Chern classes of the torsion free sheaf $\widetilde{V}$. It follows from the exact sequence (5.8) that

$$\mathrm{Ch}(\widetilde{V}) = \mathrm{Ch}(V) + \mathrm{Ch}(V_z)\,, \tag{5.10}$$

where Ch stands for the Chern character. Using the Grothendieck-Riemann-Roch theorem, we find that

$$\mathrm{Ch}(V_z) = i_*(\mathrm{Ch}(L \otimes K_{B|z})Td(z))Td(X)^{-1}\,, \tag{5.11}$$

where Td stands for the Todd class. Inserting this expression into (5.10) and expanding out, one obtains the Chern classes

$$c_1(\widetilde{V}) = c_1(V) = 0\,, \tag{5.12}$$

$$c_2(\tilde{V}) = c_2(V) + z \tag{5.13}$$

$$c_3(\tilde{V}) = c_3(V) - 2(c_1(V_z) + 1 - g) \tag{5.14}$$

where $c_1(V)$, $c_2(V)$ and $c_3(V)$ are the Chern classes of the bundle V given in (3.25), (3.26) and (3.27) respectively and g is the genus of the curve z. It follows from (5.6) that

$$c_1(V_z) = c_1(L) - c_1(B) \cdot \tilde{z}\,. \tag{5.15}$$

Using the Riemann-Roch formula on B, one can also show that the genus is given by

$$1 - g = \frac{1}{2}(c_1(B) - z) \cdot z\,. \tag{5.16}$$

Inserting these expressions into (5.14), the third Chern class of $\widetilde{V}$ can be written as

$$c_3(\widetilde{V}) = c_3(V) - 2c_1(L) + (c_1(B) + z) \cdot z \tag{5.17}$$

Note that this result is true for any allowed choice of the line bundles $\mathcal{N}$ on $\mathcal{C}$ and L on z. Having defined the torsion free sheaf $\widetilde{V}$, we can now construct its spectral line bundle $\widetilde{\mathcal{N}}$ via the inverse Fourier-Mukai transformation

$$\widetilde{V} \longrightarrow (\widetilde{\mathcal{C}}, \widetilde{\mathcal{N}})\,, \tag{5.18}$$

[4]The notion of stability here is similar to that used for vector bundles. The differential geometric counterpart of a stable sheaf $\widetilde{V}$ is a hermitean-Yang-Mills connection $\widetilde{A}$ on the vector bundle V which is smooth outside of the curve $\tilde{z} \subset X$ but has a delta function behavior along the curve $\tilde{z}$. In other words, a stable torsion free sheaf $\widetilde{V}$ is the algebraic-geometry incarnation of a "small instanton" concentrated on the curve $\tilde{z}$. This justifies the terminology "small instanton phase transition" that we use below to describe the two step process of first creating $\widetilde{V}$ out of V and then deforming $\widetilde{V}$ to a smooth vector bundle $\widehat{V}$ on X.

where $\widetilde{\mathcal{C}}$ is given in (5.3). It follows from the Fourier-Mukai transformation and (5.8) that $\widetilde{\mathcal{N}}$ must lie in the exact sequence

$$0 \to \pi^* L \to \widetilde{\mathcal{N}} \to \mathcal{N} \to 0\,, \tag{5.19}$$

where $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are understood as line bundles on $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ respectively.

This sequence implies that the line bundles $\widetilde{\mathcal{N}}$ and $\pi^* L$ on $\pi^* z$ are not independent but, rather, are related by the expression

$$\left(\widetilde{\mathcal{N}} \otimes \mathcal{O}_X(-(\mathcal{C} \cdot \pi^* z))\right)\Big|_{\pi^* z} = \pi^* L\,. \tag{5.20}$$

Evaluating the first Chern class on $\pi^* z$, we find that

$$c_1(\widetilde{\mathcal{N}})|_{\pi^* z} - \mathcal{C} \cdot \pi^* z = \pi^* c_1(L)\,. \tag{5.21}$$

We will return to this equation and a discussion of the properties of $\widetilde{\mathcal{N}}$ below.

We can now examine whether the singular torsion free sheaf, $\widetilde{V}$, can be smoothed out to a stable SU(n) vector bundle, qwhich we will denote by $\widehat{V}$. The spectral data of $\widehat{V}$ can be obtained via the inverse Fourier-Mukai transformation

$$\widehat{V} \longrightarrow (\widehat{\mathcal{C}}, \widehat{\mathcal{N}})\,. \tag{5.22}$$

Clearly, the spectral cover $\widehat{C}$ is in the homology class

$$\widehat{\mathcal{C}} = n\sigma + \pi^* \hat{\eta}\,, \tag{5.23}$$

where

$$\hat{\eta} = \eta + z\,. \tag{5.24}$$

To ensure that $\widehat{V}$ is stable we need to take $\widehat{\mathcal{C}}$ to be an irreducible surface in the class $n\sigma + \pi^* \hat{\eta}$. Furthermore, from (3.20) we see that the line bundle $\widehat{\mathcal{N}}$ must satisfy

$$c_1(\widehat{\mathcal{N}}) = n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi^* \hat{\eta} + \left(\frac{1}{2} + n\lambda\right)\pi^* c_1(B)\,. \tag{5.25}$$

The structure of $\widehat{\mathcal{N}}$ will be further discussed below.

Since $\widehat{V}$ is a smooth, stable SU(n) bundle, then it follows from (3.25), (3.26) and (3.27) that

$$c_1(\widehat{V}) = 0\,, \tag{5.26}$$

$$c_2(\widehat{V}) = \hat{\eta}\sigma - \frac{1}{24} c_1(B)^2 (n^3 - n) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right) n\hat{\eta}(\hat{\eta} - nc_1(B))\,, \tag{5.27}$$

$$c_3(\widehat{V}) = 2\lambda\sigma\hat{\eta}(\hat{\eta} - nc_1(B))\,. \tag{5.28}$$

If the bundle $\widehat{V}$ exists, then its Chern classes must match those of the torsion free sheaf $\widetilde{V}$. That is, we must have

$$c_i(\widehat{V}) = c_i(\widetilde{V})\,, \tag{5.29}$$

for $i = 1, 2, 3$. It follows from (5.26) and (5.12) that both first Chern classes vanish. Comparing the second Chern classes given in (5.27) and (5.13) respectively, we see that they will be identical if and only if one restricts the spectral line bundle $\mathcal{N}$ of the original bundle V so that

$$\lambda = \pm\frac{1}{2}\,. \tag{5.30}$$

Hence, small instanton transitions of this type only occur for certain components of the moduli space of SU(n) vector bundles. Henceforth, we will assume that (5.30) is satisfied. Inserting these values for λ into expression (5.28) for the third Chern class of $\widehat{V}$, we find, using (5.24), that it will be identical to $c_3(\widetilde{V})$ in (5.17) if and only if

$$c_1(L) = \left(\frac{1}{2}(1 \pm n)c_1(B) + \frac{1}{2}(1 \mp 1)z \mp \eta\right) \cdot z\,. \tag{5.31}$$

Therefore, the line bundle L on the curve z is also not arbitrary, but must satisfy constraint (5.31). Note that, in general, $deg(L) \in \mathbb{Z}$ but need not be positive.

If the torsion free sheaf $\widetilde{V}$ can be smoothed out to the irreducible vector bundle $\widehat{V}$, then, in addition to (5.29), the corresponding spectral line bundles must satisfy

$$c_1(\widehat{\mathcal{N}}) = c_1(\widetilde{\mathcal{N}})\,. \tag{5.32}$$

Inserting this into expression (5.21), we have

$$c_1(\widehat{\mathcal{N}})|_{\pi^* z} - \mathcal{C} \cdot \pi^* z = \pi^* c_1(L)\,. \tag{5.33}$$

Using (5.1) and (5.25), we find that

$$n\left(\lambda - \frac{1}{2}\right)\sigma \cdot \pi^* z + \pi^*\left(\left(\left(\frac{1}{2} - \lambda\right)(\eta + z) + \left(\frac{1}{2} + n\lambda\right)c_1(B) - \eta\right) \cdot z\right) = \pi^* c_1(L)\,. \tag{5.34}$$

Since the first term on the left hand side is not of the form π^* of some expression, it follows that we must take

$$\lambda = \frac{1}{2}\,, \tag{5.35}$$

for consistency. This is compatible with (5.30), but is a stronger constraint. Henceforth, we will assume that (5.35) is satisfied. In this case, expression (5.34) simplifies to

$$c_1(L) = \left(\frac{1}{2}(1 + n)c_1(B) - \eta\right) \cdot z\,, \tag{5.36}$$

which is compatible with (5.31) for the choice of $\lambda = 1/2$. It follows that the spectral line bundle L on the curve z is not only not arbitrary, but is uniquely fixed to be

$$L = \mathcal{O}_z\left(\left(\frac{1}{2}(1 + n)c_1(B) - \eta\right) \cdot z\right)\,. \tag{5.37}$$

Furthermore, note that for $\lambda = 1/2$

$$c_1(\widehat{\mathcal{N}}) = c_1(\mathcal{N}) \tag{5.38}$$

and, therefore, using (5.32) that

$$\widehat{\mathcal{N}} = \widetilde{\mathcal{N}} = \mathcal{N}\,. \tag{5.39}$$

It follows that the spectral line bundles $\widehat{\mathcal{N}}$ and $\widetilde{\mathcal{N}}$ are also uniquely fixed in terms of $\mathcal{N}$.

We conclude that for the choice of $\lambda = 1/2$ and L given by (5.37), the torsion free sheaf $\widetilde{V}$ can be smoothed out to a stable SU(n) vector bundle $\widehat{V}$ and, hence, the phase transition can be completed. Note from (5.17), (5.29) and (5.36) that

$$c_3(\widehat{V}) = c_3(V) + (2\eta + z - nc_1(B)) \cdot z \tag{5.40}$$

It follows from (4.11) that such phase transitions generically change the number of generations.

5.1 Summary

In this section we have shown the following.

- Start with a heterotic M-theory vacuum specified by a stable SU(n) vector bundle, V, on the observable boundary brane with spectral cover

  $$\mathcal{C} = n\sigma + \pi^*\eta \tag{5.41}$$

  and line bundle $\mathcal{N}$ constrained to have

  $$\lambda = \frac{1}{2} \tag{5.42}$$

  as well as a five-brane class in the bulk space

  $$W = W_B\sigma + a_f F\,, \tag{5.43}$$

  where W_B is non-vanishing. Note that

  $$W_B = \pi^* z\,, \tag{5.44}$$

  where $z = 12c_1(B) - \eta$.

- Now move the five-brane through the bulk space until it touches the observable brane and "detach" either all, or a portion, of the base component of the five-brane class. That is, consider $\pi^* z$, where z is either the entire base curve $12c_1(B) - \eta$ or some effective subcurve. Leave the rest of the base component, if any, and the pure fiber component, $a_f F$, of the five-brane class undisturbed. One can then define a rank 1 vector bundle V_z over the curve $\tilde{z} = \sigma \cdot \pi^* z$ with spectral data $(\pi^* z, \pi^* L)$ where

$$L = \left(\frac{1}{2}(1+n)c_1(B) - \eta\right) \cdot z\,. \tag{5.45}$$

- The original vector bundle, V, now combines with the rank 1 vector bundle, V_z, to form a singular torsion free sheaf, $\widetilde{V}$, on the observable brane. This sheaf has a reducible spectral cover

$$\widetilde{\mathcal{C}} = \mathcal{C} \cup \pi^* z \tag{5.46}$$

and spectral line bundle

$$\widetilde{\mathcal{N}} = \mathcal{N}\,. \tag{5.47}$$

This singular torsion free sheaf is called a small instanton.

- The small instanton can now be smoothed out into a stable $\mathrm{SU}(n)$ vector bundle $\widehat{V}$ with spectral cover

$$\widehat{\mathcal{C}} = n\sigma + \pi^*(\eta + z) \tag{5.48}$$

and spectral line bundle

$$\widehat{\mathcal{N}} = \mathcal{N}\,. \tag{5.49}$$

Note that $\widehat{V}$ has the same structure group, $\mathrm{SU}(n)$, as V and that both $\widehat{\mathcal{N}}$ and $\mathcal{N}$ have $\lambda = 1/2$.

- The Chern classes of the original vector bundle V and the final vector bundle $\widehat{V}$ after the phase transition are related by

$$c_1(\widehat{V}) = c_1(V) = 0\,, \tag{5.50}$$
$$c_2(\widehat{V}) = c_2(V) + z\,, \tag{5.51}$$
$$c_3(\widehat{V}) = c_3(V) + (2\eta + z - nc_1(B)) \cdot z\,. \tag{5.52}$$

These operations define a chirality-changing small instanton phase transition from one heterotic M-theory vacuum to another involving either all, or part, of the base component of the five-brane class. The remainder the base component, if any, and the entire pure fiber class have not been involved in this transition. In order to make these concepts more transparent, we now present several examples.

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Example 1. Consider the first sample vacuum discussed earlier in this paper, specified by $B = \mathbb{F}_1$, $G = \mathrm{SU}(5)$, spectral cover

$$\mathcal{C} = 5\sigma + \pi^*\eta\,, \tag{5.53}$$

where

$$\eta = 12\mathcal{S} + 15\mathcal{E} \tag{5.54}$$

and line bundle $\mathcal{N}$ with

$$\lambda = \frac{1}{2}\,. \tag{5.55}$$

We found in (4.7) that

$$W_B = 12\mathcal{S} + 21\mathcal{E}\,, \qquad a_f = 132 \tag{5.56}$$

and in (4.12) that

$$N_{\mathrm{gen}} = 3\,. \tag{5.57}$$

Since $\lambda = 1/2$ and $W_B \neq 0$, this vacuum satisfies the criteria to make a chirality-changing small instanton transition. To specify this, we must choose the portion of base component W_B we wish to "absorb" during the transition. Let us choose the entire base curve

$$z = 12\mathcal{S} + 21\mathcal{E} \tag{5.58}$$

In this case, the small instanton transition will be to a new, irreducible vacuum specified by $B = \mathbb{F}_1$, $G = \mathrm{SU}(5)$, spectral curve

$$\widehat{\mathcal{C}} = 5\sigma + \pi^*(\eta + z) \tag{5.59}$$

where

$$\eta + z = 24\mathcal{S} + 36\mathcal{E} \tag{5.60}$$

and line bundle $\widehat{\mathcal{N}} = \mathcal{N}$ and, hence,

$$\lambda = \frac{1}{2}\,. \tag{5.61}$$

Since we have absorbed the entire base component W_B, we must have

$$\hat{W}_B = 0\,. \tag{5.62}$$

On the other hand, the pure fiber component of the five-brane class is left undisturbed and, therefore,

$$\hat{a}_f = 132\,. \tag{5.63}$$

We note from (3.19), (3.33) and (4.8) that these are exactly the properties of the second sample vacuum that we discussed above. Furthermore, inserting $n = 5$, (5.54), (5.57) and (5.58) into (5.52) we find, using $N_{\mathrm{gen}} = \frac{1}{2}c_3$, that

$$\hat{N}_{\mathrm{gen}} = 336\,, \tag{5.64}$$

which is consistent with the result given in (4.13). We conclude that the two sample vacua discussed above are related to each other by a chirality-changing small instanton transition in which the entire base component of the five-brane class of the first vacuum is "absorbed".

Example 2. As a second example, consider a heterotic M-theory vacuum specified by $B = \mathbb{F}_0$, $G = \mathrm{SU}(3)$, spectral cover

$$\mathcal{C} = 3\sigma + \pi^*\eta \,, \tag{5.65}$$

where

$$\eta = 6\mathcal{S} + 6\mathcal{E} \tag{5.66}$$

and line bundle $\mathcal{N}$ with

$$\lambda = \frac{1}{2} \,. \tag{5.67}$$

Using the formalism presented in sections 2 and 3, it can easily be shown that

$$W_B = 18\mathcal{S} + 18\mathcal{E} \,, \qquad a_f = 100 \tag{5.68}$$

and that

$$N_{\mathrm{gen}} = 0 \,. \tag{5.69}$$

Since $\lambda = 1/2$ and $W_B \neq 0$, this vacuum satisfies the criteria to make a chirality-changing small instanton transition. To specify this, we must choose the portion of base component W_B we wish to "absorb" during the transition. In this example, we will only choose an effective subcurve of the base component

$$z = \mathcal{E} \,. \tag{5.70}$$

For this case, the small instanton transition will be to a new, irreducible vacuum specified by $B = \mathbb{F}_0$, $G = \mathrm{SU}(3)$, spectral curve

$$\widehat{\mathcal{C}} = 3\sigma + \pi^*(\eta + z) \,, \tag{5.71}$$

where

$$\eta + z = 6\mathcal{S} + 7\mathcal{E} \tag{5.72}$$

and line bundle $\widehat{\mathcal{N}} = \mathcal{N}$ and, hence,

$$\lambda = \frac{1}{2} \,. \tag{5.73}$$

Since we have absorbed π^*z where $z = \mathcal{E}$, we must have

$$\widehat{W}_B = 18\mathcal{S} + 17\mathcal{E} \,. \tag{5.74}$$

On the other hand, the pure fiber component of the five-brane class is left undisturbed and, therefore,

$$\hat{a_f} = 100\,. \tag{5.75}$$

Using (5.52) and the data presented in this example, we can compute the number of generations in the new vacuum after the phase transition. We find that

$$\hat{N}_{\text{gen}} = 3\,. \tag{5.76}$$

We conclude that, in this example, we have a chirality-changing small instanton transition from a vacuum with no families to a vacuum with three families. The three family vacuum has both non-vanishing base and fiber components of its five-brane class.

6. Gauge group changing small instanton transitions

In the previous section, we discussed chirality-changing phase transitions involving all, or a portion, of the base component, W_B, of the five-brane class, W. This discussion did not include the pure fiber component, $a_f F$, of W, which was left "unabsorbed" by the transition. In this section, we turn our attention to the pure fiber component. We show that all, or a portion, of $a_f F$ can always be "absorbed" via a small instanton phase transition into a smooth vector bundle on the observable brane. This vector bundle, however, is somewhat different than the stable $\mathrm{SU}(n)$ bundles discussed previously. Among other properties, it is a reducible, semi-stable vector bundle and has the product structure group $\mathrm{SU}(n) \times \mathrm{SU}(m)$. It follows that pure fiber component small instanton transitions generically change the structure group and, hence, the gauge group on the observable brane.

6.1 Reducible vector bundles

We begin by considering a stable $\mathrm{SU}(n)$ vector bundle, $\overline{V}$, over X specified by a spectral cover

$$\bar{\mathcal{C}} = n\sigma + \pi^*\bar{\eta} \tag{6.1}$$

and line bundle $\overline{\mathcal{N}}$ over $\bar{\mathcal{C}}$. The Chern classes $c_1(\overline{\mathcal{N}})$ and $c_i(\overline{V})$ for $i = 1, 2, 3$ are given in with η replaced by $\bar{\eta}$.

Next, let M be a stable $\mathrm{SU}(m)$ vector bundle with $m \geq 2$, not over X, but over the base B. The Chern classes of M are trivial to compute and are given by

$$c_1(M) = 0 \tag{6.2}$$

$$c_2(M) = k\,, \qquad k \in \mathbb{Z}\,, \tag{6.3}$$

$$c_i(M) = 0\,, \qquad i \geq 3\,. \tag{6.4}$$

Now consider the pull-back to a stable SU(m) vector bundle π^*M over X. The spectral data can be determined by performing an inverse Fourier-Mukai transformation. The result is that

$$\pi^*M \longrightarrow (m\sigma, M)\,, \tag{6.5}$$

where the spectral cover $m\sigma$ consists of m coincident sections, called a non-reduced surface [12]. Although of rank m, π^*M splits into a direct sum of m one-dimensional spaces over each point in the base B. Hence, M can be thought of as a deformation of a line bundle over $m\sigma$. The Chern classes of π^*M follow directly from the above and are given by

$$c_1(\pi^*M) = 0 \tag{6.6}$$
$$c_2(\pi^*M) = kF\,, \qquad k \in \mathbb{Z}\,, \tag{6.7}$$
$$c_3(\pi^*M) = 0\,. \tag{6.8}$$

Having presented the two stable vector bundles $\overline{V}$ and π^*M, we now construct a reducible bundle, $\mathbf{V}$, by taking their direct sum. That is, define

$$\mathbf{V} = \overline{V} \oplus \pi^*M \tag{6.9}$$

$\mathbf{V}$ is a smooth, but reducible, semi-stable, rank $n+m$ vector bundle over X with structure group SU(n) $\times$ SU(m). Its spectral data can be computed via an inverse Fourier-Mukai transformation with the result that

$$\mathbf{V} \longrightarrow (\mathbf{C}, \mathbf{N})\,, \tag{6.10}$$

where

$$\mathbf{C} = \bar{\mathcal{C}} \cup m\sigma, \qquad \mathbf{N} = \overline{\mathcal{N}} \oplus \sigma_* M\,. \tag{6.11}$$

Since

$$c_1(\overline{V}) = c_1(\pi^*M) = 0\,, \tag{6.12}$$

the Chern classes of $\mathbf{V}$ are simply the sum

$$c_i(\mathbf{V}) = c_i(\overline{V}) + c_i(\pi^*M)\,, \tag{6.13}$$

for $i = 1, 2, 3$. It follows that

$$c_1(\mathbf{V}) = 0\,, \tag{6.14}$$
$$c_2(\mathbf{V}) = c_2(\overline{V}) + kF\,, \qquad k \in \mathbb{Z}\,, \tag{6.15}$$
$$c_3(\mathbf{V}) = c_3(\overline{V})\,. \tag{6.16}$$

This concludes our discussion of reducible SU(n) $\times$ SU(m) vector bundles. We now turn to the study of small instanton phase transitions involving the pure fiber component of the five-brane class.

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Let us begin on the observable boundary brane with a stable $\mathrm{SU}(n)$ vector bundle V specified by the spectral cover

$$\mathcal{C} = n\sigma + \pi^*\eta \tag{6.17}$$

and the line bundle $\mathcal{N}$ over $\mathcal{C}$ whose first Chern class satisfies (3.20). It follows from the above discussion that the bulk space five-brane class $W = W_B\sigma + a_f F$ does not vanish. In addition, we will demand that V and TX be such that the fiber component

$$a_f F \neq 0\,. \tag{6.18}$$

We will will make no assumptions about the base component, W_B, which can be either zero or non-zero, but does not concern us in this section.

Let us now move the bulk five-brane to the observable boundary brane and attempt to "absorb" the kF part of the five-brane class into the vector bundle. A full gauge-changing phase transition will occur if we choose $k = a_f$. However, a "partial" transition can occur for $k < a_f$. Our discussion will be similar to that of the previous section, with the important difference that we must first consider vector bundles over the base B before lifting them to X. With this in mind, define a rank m vector bundle , U, over B by

$$U = \mathcal{O}_B \oplus \ldots \oplus \mathcal{O}_B\,, \tag{6.19}$$

with m factors of the trivial bundle $\mathcal{O}_B$ over the base. Henceforth, for simplicity, we will consider the generic region of moduli space where the class kF is represented by k separated fibers. Projected onto the base, this corresponds to k distinct points, z_i, with $i = 1, \ldots, k$. As in the construction of $\widetilde{V}$ from V and V_z in section 5, we should next specify a line bundle over these points. This is accomplished by choosing, at each point z_i, a one dimensional vector space U_{z_i}. The space

$$U_z = U_{z_1} \cup \cdots \cup U_{z_k} \tag{6.20}$$

then defines a line bundle over the base $B_z = \{z_1, \ldots, z_k\}$ of points. Given these two separate vector bundles, we now attempt to "weave" them together by relating them where they overlap, namely, on B_z. This is done by specifying a surjection

$$\xi : U|_{z_i} \to U_{z_i}\,. \tag{6.21}$$

That such a surjection exists will become clear shortly. Given this relation, one can define a "bundle" $\widetilde{U}$ via the exact sequence

$$0 \to \widetilde{U} \to U \to U_z \to 0\,. \tag{6.22}$$

It can be shown that because

$$\text{codimension}\, B_z = 2 > 1\,, \tag{6.23}$$

then $\widetilde{U}$ is a singular torsion free sheaf, but is not a smooth vector bundle. It is easy to show that the general smooth bundle obtained as a deformation of $\widetilde{U}$ will be stable if and only if

$$2 \leq m\,. \tag{6.24}$$

Henceforth, we will assume this condition is satisfied. It is straightforward to compute the Chern classes of $\widetilde{U}$. They are given by

$$c_1(\widetilde{U}) = 0\,, \tag{6.25}$$

$$c_2(\widetilde{U}) = k\,, \tag{6.26}$$

$$c_i(\widetilde{U}) = 0\,, \qquad i \geq 3\,. \tag{6.27}$$

We now want to construct the spectral data for the above vector bundles and sheaf via the inverse Fourier-Mukai transformation. We emphasize that these transformations are to be carried out in the base space B. For simplicity, we will assume that $B = dP_9$. The space dP_9 is an elliptic fibration over $\mathbb{CP}^1$ with $\pi_B : dP_9 \to \mathbb{CP}^1$ and admits a zero section σ_B. Our conclusions, however, can be shown to hold generically for any other allowed base B [21]. Performing the Fourier-Mukai transformations, we find that

$$U \longrightarrow (m\sigma_B, \mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{CP}^1}(-1))\,, \tag{6.28}$$

with m factors of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ and

$$U_z \longrightarrow \bigoplus_{i=1}^{k}(f_i, \mathcal{O}_{f_i}(z_i - p_i))\,, \tag{6.29}$$

where $f_i = \pi_B^{-1}(\pi_B(z_i))$ is the elliptic fiber containing the point z_i and $p_i = \sigma_B(\pi_B(z_i))$ is the origin of f_i. In addition, we have

$$\widetilde{U} \longrightarrow (m\sigma_B + kf, \widetilde{\mathcal{N}}_B)\,, \tag{6.30}$$

where $f \in H_2(B, \mathbb{Z})$ denotes the class of the fiber of the projection $\pi_B : B \to \mathbb{CP}^1$. Now, it follows from the Fourier-Mukai transformations and (6.22) that $\widetilde{\mathcal{N}}_B$ must lie in the exact sequence

$$0 \to \bigoplus_{i=1}^{k} \mathcal{O}_{f_i}(z_i - p_i) \to \widetilde{\mathcal{N}}_B \to \mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{CP}^1}(-1) \to 0\,. \tag{6.31}$$

We see that $\widetilde{\mathcal{N}}_B$ must satisfy the condition that

$$\left(\widetilde{\mathcal{N}}_B \otimes \mathcal{O}_B(-m\sigma_B)\right)\Big|_{f_i} = \mathcal{O}_{f_i}(z_i - p_i)\,. \tag{6.32}$$

Calculating the first Chern classe of this expression, we find

$$c_1(\widetilde{\mathcal{N}}_B)\Big|_{f_i} = m\,, \tag{6.33}$$

for each $i = 1, \ldots, k$.

We can now examine whether the singular torsion free sheaf, $\widetilde{U}$, can be smoothed out to a stable SU(m) vector bundle on the base, which we will denote by $\widehat{U}$. The spectral data of $\widehat{U}$ can be obtained via the inverse Fourier-Mukai transformation

$$\widehat{U} \longrightarrow (\widehat{\mathcal{C}}_B, \widehat{\mathcal{N}}_B)\,. \tag{6.34}$$

To ensure the stability of the bundle $\widehat{U}$, it suffices to choose $\widehat{\mathcal{C}}_B$ to be an irreducible curve in the homology class

$$\widehat{\mathcal{C}}_B = m\sigma_B + kf\,. \tag{6.35}$$

Such a homology class does not necessarily contain an irreducible curve. It will contain such a curve if and only if

$$1 < m \leq k\,, \tag{6.36}$$

which is an important constraint on the choice of m. We, henceforth, assume this condition is satisfied. We can also show that the spectral line bundle, $\widehat{\mathcal{N}}_B$, on $\widehat{\mathcal{C}}$ must satisfy

$$c_1(\widehat{\mathcal{N}}_B) = \frac{m}{2}(2k - 1 - m)\,. \tag{6.37}$$

The Chern classes of the stable SU(m) bundle $\widehat{U}$ are easily computed. We find that

$$c_1(\widehat{U}) = 0\,, \tag{6.38}$$

$$c_2(\widehat{U}) = k\,, \tag{6.39}$$

$$c_i(\widehat{U}) = 0\,, \qquad i \geq 3\,, \tag{6.40}$$

which are identical to the Chern classes of the torsion free sheaf, $\widetilde{U}$, given in (6.25), (6.26) and (6.27). It follows that, as far as these Chern classes are concerned, the torsion free sheaf, $\widetilde{U}$, can be smoothed out to the stable vector bundle, $\widehat{U}$, without any further restrictions. However, it remains to check whether the corresponding spectral line bundles satisfy

$$c_1(\widehat{\mathcal{N}}_B) = c_1(\widetilde{\mathcal{N}}_B)\,. \tag{6.41}$$

In contrast with the discussion in section 5, this condition does not impose additional constraints. In fact, one can check that, on the reducible spectral curve $\widetilde{\mathcal{C}}$, the sheaf $\widetilde{N}_B$ can be deformed to a line bundle satisfying, in addition to (6.33), the condition

$$c_1(\widetilde{\mathcal{N}}_B) = \frac{m}{2}(2k - 1 - m)\,. \tag{6.42}$$

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Hence, it follows from (6.37) and (6.42) that sheaf, $\widetilde{U}$, can always be smoothed out to the stable SU(m) vector bundle $\widehat{U}$.

Having defined these bundles and sheafs on the base B, we can now pull them all back to X. In particular, $\pi^*\widetilde{U}$ is defined on X and, since

$$\text{codimension}\, kF = 2 > 1\,, \tag{6.43}$$

it follows that $\pi^*\tilde{U}$ is a singular, torsion free sheaf. The Chern classes of $\pi^*\tilde{U}$ in X are simply the pull-back of the Chern classes of $\tilde{U}$ in the base. They are given by

$$c_1(\pi^*\tilde{U}) = 0\,, \tag{6.44}$$
$$c_2(\pi^*\tilde{U}) = kF\,, \tag{6.45}$$
$$c_3(\pi^*\tilde{U}) = 0\,. \tag{6.46}$$

In addition, it follows from (6.24) and (6.28) that for

$$2 \leq m \leq k \tag{6.47}$$

the pull-back $\pi^*\widehat{U}$ is a stable SU(m) vector bundle on X with spectral cover

$$\hat{\mathcal{C}} = m\sigma \tag{6.48}$$

and $\widehat{\mathcal{N}} = \widehat{U}$, which can be thought as a deformation of a line bundle over $m\sigma$. The Chern classes of $\pi^*\hat{U}$ in X are simply the pull-back of the Chern classes of $\hat{U}$ in the base. They are given by

$$c_1(\pi^*\hat{U}) = 0\,, \tag{6.49}$$
$$c_2(\pi^*\hat{U}) = kF\,, \tag{6.50}$$
$$c_3(\pi^*\hat{U}) = 0\,. \tag{6.51}$$

Finally, it is not hard to establish that, since $\tilde{U}$ can be smoothed out to $\hat{U}$ in the base, the pull-back torsion free sheaf, $\pi^*\widetilde{U}$, can be smoothed out to vector bundle $\pi^*\hat{U}$ on the Calabi-Yau threefold X.

The small instanton transition then proceeds as follows. Move the bulk five-brane to the observable boundary brane. The pure fiber component, kF of the five-brane class combines with the original stable vector bundle V on the boundary brane to form a reducible, singular, torsion free sheaf

$$\widetilde{\mathbf{V}} = V \oplus \pi^*\tilde{U}\,. \tag{6.52}$$

This small instanton can then be smoothed out to a reducible, but smooth, semi-stable SU$(n)\times$SU(m) vector bundle

$$\widehat{\mathbf{V}} = V \oplus \pi^*\hat{U}\,. \tag{6.53}$$

It follows from (6.13) and (6.49), (6.50), (6.51) that the Chern classes of $\widehat{\mathbf{V}}$ are given by

$$c_1(\widehat{\mathbf{V}}) = 0\,, \tag{6.54}$$
$$c_2(\widehat{\mathbf{V}}) = c_2(V) + kF\,, \tag{6.55}$$
$$c_3(\widehat{\mathbf{V}}) = c_3(V)\,. \tag{6.56}$$

Note that this phase transition changes the structure group on the boundary brane from $\mathrm{SU}(n)$ to $\mathrm{SU}(n)\times\mathrm{SU}(m)$ where $2 \leq m \leq k$ and, hence, their commutant subgroups in E_8 also change. We conclude that such small instanton phase transitions generically change the unbroken gauge group on the boundary brane.

6.2 Summary

In this section we have shown the following.

- Start with a heterotic M-theory vacuum specified by a stable $\mathrm{SU}(n)$ vector bundle, V, on the observable boundary brane with spectral cover
$$\mathcal{C} = n\sigma + \pi^*\eta \tag{6.57}$$
and line bundle $\mathcal{N}$ over $\mathcal{C}$ satisfying (3.20), as well as a five-brane class in the bulk space
$$W = W_B\sigma + a_f F\,, \tag{6.58}$$
where $a_f F$ is non-vanishing.

- Now move the five-brane through the bulk space until it touches the observable brane and "detach" either all, or a portion, of the fiber component of the five-brane class. That is, consider kF, where kF is either the entire fiber class, for $k = a_f$, or some effective subclass, for $k < a_f$. Leave the pure base component, W_B if any, and the rest of the fiber component, if any, of the five-brane class undisturbed. One can then define a singular torsion free sheaf $\pi^*\widetilde{U}$ over X associated with kF.

- The original vector bundle, V, now combines with $\pi^*\widetilde{U}$, to form a reducible, singular, torsion free sheaf
$$\widetilde{\mathbf{V}} = V \oplus \pi^*\tilde{U} \tag{6.59}$$
on the observable brane. This singular torsion free sheaf is called a small instanton.

- The small instanton can now be smoothed out into a reducible, semi-stable $\mathrm{SU}(n)\times\mathrm{SU}(m)$ vector bundle
$$\widehat{\mathbf{V}} = V \oplus \pi^*\hat{U}\,, \tag{6.60}$$
where m can be any integer subject to the constraint $2 \leq m \leq k$.

- The Chern classes of the original vector bundle V and the final bundle $\widehat{\mathbf{V}}$ after the phase transition are related by

$$c_1(\widehat{\mathbf{V}}) = c_1(V) = 0\,, \tag{6.61}$$
$$c_2(\widehat{\mathbf{V}}) = c_2(V) + kF\,, \tag{6.62}$$
$$c_3(\widehat{\mathbf{V}}) = c_3(V)\,. \tag{6.63}$$

- The structure group of the vector bundle changes during the phase transition from

$$\mathrm{SU}(n) \longrightarrow \mathrm{SU}(n) \times \mathrm{SU}(m)\,, \tag{6.64}$$

where $2 \leq m \leq k$. It follows that the unbroken gauge group on the boundary brane, the commutant in E_8 of the structure group, also undergoes a transition.

These operations define a gauge-changing small instanton phase transition from one heterotic M-theory vacuum to another involving either all, or part, of the fiber component of the five-brane class. The base component, if any, and the remainder of the pure fiber class, if any, have not been involved in this transition. In order to make these concepts more transparent, we now present an example.

Example. Consider a vacuum specified by $B = \mathbb{F}_1$, $G = \mathrm{SU}(5)$, the irreducible spectral curve

$$\mathcal{C} = 5\sigma + \pi^*\eta\,, \qquad \eta = 24\mathcal{S} + 36\mathcal{E} \tag{6.65}$$

and line bundle $\mathcal{N}$ with $\lambda = 1/2$. We showed in section 5 that the associated five-brane class is given by

$$W_B = 0\,, \qquad a_f = 132 \tag{6.66}$$

and that the number of generations is

$$N_{\mathrm{gen}} = 336\,. \tag{6.67}$$

Note that the commutant of $G = \mathrm{SU}(5)$ in E_8 is the unbroken gauge group

$$H = \mathrm{SU}(5)\,. \tag{6.68}$$

Since $a_f \neq 0$, this vacuum satisfies the criterion to make a gauge-changing small instanton phase transition. To specify this, we must choose the portion of the fiber component, kF, that we want to "absorb" during the transition. Let us choose the entire fiber class by taking

$$k = 132\,. \tag{6.69}$$

In this case, the small instanton transition will be to a new smooth, but reducible, semi-stable vacuum specified by $B = \mathbb{F}_1$ and

$$\widehat{G} = \mathrm{SU}(5) \times \mathrm{SU}(m)\,, \qquad 2 \leq m \leq 132\,. \tag{6.70}$$

The entire five-brane class has been absorbed, so

$$\widehat{W}_B = 0\,, \qquad \hat{a}_f = 0\,. \tag{6.71}$$

Since the third Chern class does not change during the transition, it follows that

$$\widehat{N}_{\mathrm{gen}} = 336\,. \tag{6.72}$$

By construction, $\mathrm{SU}(m)$ must commute with the structure group $G = \mathrm{SU}(5)$. This implies that

$$\mathrm{SU}(m) \subseteq H = \mathrm{SU}(5) \tag{6.73}$$

and, hence, that m is further restricted to satisfy $2 \leq m \leq 5$. Moreover, it follows that the commutant, $\widehat{H}$, of $\mathrm{SU}(5) \times \mathrm{SU}(m)$ in E_8 is the same as the commutant of $\mathrm{SU}(m)$ in $H = \mathrm{SU}(5)$. It is helpful to note that the maximal subgroups of $\mathrm{SU}(5)$ containing an $\mathrm{SU}(m)$ factor are

$$\mathrm{SU}(5) \supset \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)\,, \qquad \mathrm{SU}(4) \times \mathrm{U}(1)\,. \tag{6.74}$$

Let us first consider $m = 2$. Using (6.74), we see that

$$\widehat{H} = \mathrm{SU}(3) \times \mathrm{U}(1)\,. \tag{6.75}$$

That is, the small instanton phase transition has changed the gauge group on the boundary brane from

$$\mathrm{SU}(5) \longrightarrow \mathrm{SU}(3) \times \mathrm{U}(1)\,. \tag{6.76}$$

Similarly, for $m = 3$ the gauge group on the boundary brane undergoes the transition

$$\mathrm{SU}(5) \longrightarrow \mathrm{SU}(2) \times \mathrm{U}(1)\,, \tag{6.77}$$

whereas for $m = 4$

$$\mathrm{SU}(5) \longrightarrow \mathrm{U}(1)\,. \tag{6.78}$$

Finally, we see from (6.74) that for $m = 5$ the gauge group on the boundary brane changes as

$$\mathrm{SU}(5) \longrightarrow \mathbf{1}\,. \tag{6.79}$$

This example highlights two additional properties of gauge-changing small instanton phase transitions. The first concerns the evaluation of the final unbroken gauge group. Consider any gauge-changing phase transition in which the structure group of the vector bundle changes as

$$\mathrm{SU}(n) \longrightarrow \mathrm{SU}(n) \times \mathrm{SU}(m)\,, \tag{6.80}$$

where $2 \leq m \leq k$ and $k \leq a_f$. Let H be the commutant of $\mathrm{SU}(n)$ in E_8 and $\widehat{H}$ be the commutant of $\mathrm{SU}(n) \times \mathrm{SU}(m)$ in E_8. Clearly, since $\mathrm{SU}(m)$ must commute with $\mathrm{SU}(n)$ we have

$$\mathrm{SU}(m) \subseteq H\,. \tag{6.81}$$

As we learned in the example, this fact generically puts a much stronger restriction on m. The exact restriction depends on the choice of the structure group $\mathrm{SU}(n)$. In the above example, it tightened the bound on m from $2 \le m \le 132$ to $2 \le m \le 5$. Furthermore, it follows from (6.81) that $\widehat{H}$ must also be the commutant of $\mathrm{SU}(m)$ in H. This observation facilitates the evaluation of $\widehat{H}$ considerably. For example, let the original stable vector bundle have structure group

$$G = \mathrm{SU}(4)\,. \tag{6.82}$$

Since $\mathrm{SU}(4) \times \mathrm{SO}(10) \subset E_8$ is a maximal subgroup, it follows that

$$H = \mathrm{SO}(10)\,. \tag{6.83}$$

Now consider a small instanton transition to a reducible vector bundle with the structure group

$$\widehat{G} = \mathrm{SU}(4) \times \mathrm{SU}(m)\,, \tag{6.84}$$

where, in general, $2 \le m \le k$ for some $k \le a_f$. We see from (6.81), however, that

$$\mathrm{SU}(m) \subseteq \mathrm{SO}(10)\,. \tag{6.85}$$

It is helpful to note that

$$\mathrm{SO}(10) \supset \mathrm{SU}(2) \times \mathrm{SO}(7)\,, \qquad \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)\,, \qquad \mathrm{SU}(5) \times \mathrm{U}(1) \tag{6.86}$$

are the maximal subgroups of $\mathrm{SO}(10)$ containing an $\mathrm{SU}(m)$ factor. It follows that m is further constrained to satisfy

$$2 \le m \le 5\,. \tag{6.87}$$

Let us first consider the $m = 2$ case. Using (6.86), we see that

$$\widehat{H} = \mathrm{SO}(7)\,, \qquad \mathrm{SU}(2) \times \mathrm{SU}(4) \tag{6.88}$$

depending upon the embedding of $\mathrm{SU}(2)$ in $\mathrm{SO}(10)$. That is, the small instanton phase transition has changed the gauge group from

$$\mathrm{SO}(10) \longrightarrow \mathrm{SO}(7)\,, \qquad \mathrm{SU}(2) \times \mathrm{SU}(4)\,. \tag{6.89}$$

Clearly for the $m = 3$ case

$$\widehat{H} = \mathbf{1} \Longrightarrow \mathrm{SO}(10) \longrightarrow \mathbf{1}\,. \tag{6.90}$$

Similarly, for $m = 4$

$$\widehat{H} = \mathrm{SU}(2) \times \mathrm{SU}(2) \Longrightarrow \mathrm{SO}(10) \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{6.91}$$

and for the $m = 5$ case

$$\widehat{H} = \mathrm{U}(1) \Longrightarrow \mathrm{SO}(10) \longrightarrow \mathrm{U}(1)\,. \tag{6.92}$$

We conclude that the gauge bgroup breaking pattern for a gauge-changing small instanton phase transition can be computed and is, generically, very rich.

This observation leads to the second issue regarding gauge-group changing phase transitions. That is, given the initial data of the stable vector bundle and the associated five-brane class, can one predict which region of moduli space the vacuum is in after the phase transition? In particular, can one predict the final structure group factor $\mathrm{SU}(m)$? The answer, for the moment, must be no. The reason is that, as the five-brane "collides" with the boundary brane, tensionless string states are momentarily created due to the vanishing tension of wrapped membranes stretched between the boundary brane and the wrapped five-brane. These states become massive as the small instanton is smoothed out and can, hence, be ignored after the phase transition. However, they make it difficult to follow the moduli space trajectory of the vacuum during the transition itself. We conclude that, presently, we must be content with constructing the moduli space and specifying the gauge group breaking patterns of gauge-changing small instanton phase transitions.

7. Conclusion

In this paper, we have given a detailed description of both the mathematics and physics of small instanton phase transitions associated with the "collision" of a bulk space five-brane with a boundary brane. We expect such collisions and, hence, small instanton phase transitions to be an important part of any realistic particle physics theory derived from the "brane world". We have presented our results within the context of the fundamental brane world that arises from M-theory, namely, heterotic M-theory [4, 5, 6]. However, our results, with relatively minor modifications, are applicable to any brane world scenario [26, 27, 28].

Specifically, we have shown that upon collision with a boundary brane, part, or all, of the five-brane is "absorbed" by the boundary brane, depending upon the initial vector bundle and five-brane data. The absorbed five-brane is transmuted, rather catastrophically, into a singular modification of the initial vector bundle on the boundary brane, called a small instanton. This is then smoothed out to a modified vector bundle that differs quantitatively from the vector bundle prior to the collision. That is, the five-brane collides with the boundary brane and disappears, but at the cost of modifying the vector bundle on the boundary brane. In this paper, we have given a precise description of these small instanton phase transitions.

First, we have shown that if all, or a part, of the base component of the five-brane class is absorbed during the collision, then the vector bundle on the boundary brane is modified in such a way that its third Chern class changes. This implies that the number of generations of quarks and leptons is different after the phase transition than it was before. Specifically, we find that

$$N_{\rm gen}(\widehat{V}) = N_{\rm gen}(V) + \frac{1}{2}(2\eta + z - nc_1(B)) \cdot z \tag{7.1}$$

where V and $\widehat{V}$ are the vector bundles on the boundary brane before and after the transition respectively. Curve η and n are initial data for the vector bundle on the boundary brane, curve z specifies how much of the five-brane class is absorbed during the transition and $c_1(B)$ partially defines the Calabi-Yau vacuum. The point is that one can specify this data mathematically and explicitly compute the difference in the number of generations. We showed that the structure group of the vector bundle does not change during such phase transitions. For these reasons, we call transitions that involve only the base component of the five-brane class "chirality-changing small instanton phase transitions". At least for the types of vector bundles discussed in this paper, we find that only for specific initial vector bundle data, namely $\lambda = 1/2$, can chirality-changing transitions proceed. In all other cases they are topologically obstructed.

Second, we have shown that if all, or part, of the fiber component of the five-brane class is absorbed during the collision, then the vector bundle on the boundary brane is modified in such a way that the structure group changes. This implies that the unbroken gauge group on the boundary brane, the commutant of the structure group in E_8, is changed by the phase transition. Specifically, we find that

$$\mathrm{SU}(n) \longrightarrow \mathrm{SU}(n) \times \mathrm{SU}(m)\,, \tag{7.2}$$

where $\mathrm{SU}(n)$ and $\mathrm{SU}(n) \times \mathrm{SU}(m)$ are the structure groups before and after the phase transition respectively. The values of m are not fixed, but are constrained to lie in a relatively small interval that can be computed explicitly. Thus, we can quantitatively compute the unbroken gauge group structure following the small instanton transition. We showed that the third Chern class of the vector bundle does not change during such transitions. For these reasons, we call transitions involving only the fiber component of the five-brane class "gauge-changing small instanton phase transitions". Unlike the case of chirality-changing transitions, we find that gauge-changing phase transitions are never topologically obstructed and can always occur.

In general, small instanton phase transitions involving both the base and fiber components of the five-brane class can occur. These will then, of course, involve phase transitions in both the number of chiral families and the unbroken gauge group. There would appear to be an enormous amount of new, non-perturbative particle physics and cosmology associated with the small instanton phase transitions discussed in this paper. We will discuss these topics elsewhere.

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In this paper, we did not discuss duality between the vacua of heterotic M-theory and other theories, such as F-theory. We want to point out, however, that there is an interesting dual process in F-theory which follows from the results in our paper. The dual of heterotic M-theory on an Calabi-Yau elliptically fibered threefold with bundle data is F-theory compactified on an elliptic Calabi-Yau fourfold. We considered chirality changing phase transitions in section 5. In [29], it is shown that the third Chern class of the vector bundle is related to the four-form flux on the F-theory side. In type-IIB language, this flux is the nontrivial NS-NS and R-R three-form field background [30, 31]. Since we change the third Chern class of the vector bundle in chirality changing small instanton phase transitions, the dual process in F-theory changes this flux. On the heterotic M-theory side, the number of five-branes wrapping on a fiber does not change. The dual of the 5-branes wrapping on a fiber are the three-branes in F-theory [32]. Hence, the number of three-branes does not change in the dual process in F-theory. The anomaly equation in F-theory tells us that the sum of the number of three-branes and the flux is proportional to the Euler character of the fourfold. The dual process of the small instanton transition involving five-branes wrapping on a base curve changes the topology of the dual elliptic fourfold in F-theory, since it involves blowdown and blowup processes [20, 33, 34]. Hence, the Euler character of the manifold does change before and after the dual process. Thus, we can conclude that the F-theory dual process mediates between different manifolds with different flux. This is difficult to check directly in F-theory, since vacua with flux are not well understood. In a similar fashion, by further exploration of heterotic M-theory on an elliptic Calabi-Yau threefold, we can indirectly learn many interesting facts about dual vacua, which are worthy of further investigation.

Acknowledgments

We would like to thank E. Diaconescu, R. Donagi, A. Grassi, K. Intriligator, G. Rajesh and E. Witten for useful conversation and discussion. B.A. Ovrut is supported in part by a Senior Alexander von Humboldt Award, by the DOE under contract No. DE-AC02-76-ER-03071 and by a University of Pennsylvania Research Foundation Grant. T. Pantev is supported in part by an NSF grant DMS-9800790 and by an Alfred P. Sloan Research Fellowship. J. Park is supported in part by U.S. Department of Energy Grant No. DE-FG02-90-ER40542.

References

[1] P. Hořava and E. Witten, *Heterotic and type-I string dynamics from eleven dimensions*, *Nucl. Phys.* **B 460** (1996) 506 [hep-th/9510209].

[2] P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B 475** (1996) 94 [hep-th/9603142].

[3] E. Witten, *Strong coupling expansion of Calabi-Yau compactification*, *Nucl. Phys.* **B 471** (1996) 135 [hep-th/9602070].

[4] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, *The universe as a domain wall*, *Phys. Rev.* **D 59** (1999) 086001 [hep-th/9803235].

[5] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, *Heterotic M-theory in five dimensions*, *Nucl. Phys.* **B 552** (1999) 246 [hep-th/9806051].

[6] A. Lukas, B.A. Ovrut and D. Waldram, *Non-standard embedding and five-branes in heterotic M-theory*, *Phys. Rev.* **D 59** (1999) 106005 [hep-th/9808101].

[7] S. Donaldson, *Proc. London Math. Soc.* **3** (1985) 1.

[8] K. Uhlenbeck and S.-T. Yau, *Comm. Pure App. Math.* **39** (1986) 257; *Comm. Pure App. Math.* **42** (1986) 703.

[9] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Comm. Math. Phys.* **187** (1997) 679 [hep-th/9701162].

[10] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, alg-geom/9709029.

[11] R. Donagi, *Asian J. Math.* **1** (1997) 214.

[12] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, *On four-dimensional compactifications of F-theory*, *Nucl. Phys.* **B 505** (1997) 165 [hep-th/9701165].

[13] R. Donagi, A. Lukas, B.A. Ovrut and D. Waldram, *Non-perturbative vacua and particle physics in M-theory*, *J. High Energy Phys.* **05** (1999) 018 [hep-th/9811168].

[14] R. Donagi, A. Lukas, B.A. Ovrut and D. Waldram, *Holomorphic vector bundles and non-perturbative vacua in M-theory*, *J. High Energy Phys.* **06** (1999) 034 [hep-th/9901009].

[15] R. Donagi, B.A. Ovrut, T. Pantev and D. Waldram, *Standard models from heterotic M-theory*, hep-th/9912208.

[16] T. Banks and M. Dine, *Couplings and scales in strongly coupled heterotic string theory*, *Nucl. Phys.* **B 479** (1996) 173 [hep-th/9605136];
I. Antoniadis and M. Quiros, *Large radii and string unification*, *Phys. Lett.* **B 392** (1997) 61 [hep-th/9609209];
K. Benakli, *Phenomenology of low quantum gravity scale models*, *Phys. Rev.* **D 60** (1999) 104002 [hep-ph/9809582].

[17] E. Witten, *Small instantons in string theory*, *Nucl. Phys.* **B 460** (1996) 541 [hep-th/9511030].

[18] S. Kachru and E. Silverstein, *Chirality-changing phase transitions in 4d string vacua*, *Nucl. Phys.* **B 504** (1997) 272 [hep-th/9704185].

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[19] A. Grassi, *Internat. J. Math.* **4** (1993) 203.

[20] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds, 2*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161].

[21] B.A. Ovrut, T. Pantev and J. Park, in preparation.

[22] P.S. Aspinwall and R.Y. Donagi, *The heterotic string, the tangent bundle and derived categories*, *Adv. Theor. Math. Phys.* **2** (1998) 1041 [hep-th/9806094].

[23] G. Curio, *Chiral matter and transitions in heterotic string models*, *Phys. Lett.* **B 435** (1998) 39 [hep-th/9803224].

[24] Z. Lalak, S. Pokorski and S. Thomas, *Beyond the standard embedding in M-theory on $S^1/\mathbb{Z}_2$*, *Nucl. Phys.* **B 549** (1999) 63 [hep-ph/9807503].

[25] M. Green, J. Schwarz and E. Witten, *Superstring theory, 2*, Cambridge University Press, Cambridge 1987.

[26] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370 [hep-ph/9905221]; *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [hep-th/9906064].

[27] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *The hierarchy problem and new dimensions at a millimeter*, *Phys. Lett.* **B 429** (1998) 263 [hep-ph/9803315];
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *New dimensions at a millimeter to a fermi and superstrings at a TeV*, *Phys. Lett.* **B 436** (1998) 257 [hep-ph/9804398].

[28] Z. Kakushadze and S.H. H. Tye, *Brane world*, *Nucl. Phys.* **B 548** (1999) 180 [hep-th/9809147].

[29] B. Andreas and G. Curio, *On discrete twist and four-flux in $N = 1$ heterotic/F-theory compactifications*, hep-th/9908193.

[30] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, hep-th/9906070.

[31] S. Sethi and M. Stern, *Supersymmetry and the Yang-Mills effective action at finite N*, *J. High Energy Phys.* **06** (1999) 004 [hep-th/9903049].

[32] S. Sethi, C. Vafa and E. Witten, *Constraints on low-dimensional string compactifications*, *Nucl. Phys.* **B 480** (1996) 213 [hep-th/9606122].

[33] D.-E. Diaconescu and G. Rajesh, *Geometrical aspects of five-branes in heterotic/F-theory duality in four dimensions*, *J. High Energy Phys.* **06** (1999) 002 [hep-th/9903104].

[34] E. Witten, *Phase transitions in M-Theory and F-theory*, *Nucl. Phys.* **B 471** (1996) 195 [hep-th/9603150].